Reflected Brownian motion on nested fractals

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Motivation

K. Pietruska-Paluba in the paper The Lifechtz singularity for the density of states on the Sierpinski gasket (1991) proved the existence of the density of states for the Laplacian on the infinite Sierpinski gasket. To obtain an upper-bound for the Laplace transform of the density of states, one must reduce the problem to the one on compact space-state (the single fractal complex). We do this by reflecting the Brownian motion on vertices of fractal complexes. The estimates for unbounded fractal are obtained when we pass to infinity with the diameter of the complex.

Notation

For \( d \geq 1 \) let \( \Psi_i : \mathbb{R}^d \to \mathbb{R}^d, 1 \leq i \leq M \) be similitudes given by formula

\[ \Psi_i(x) = (1/d_i^2) L_i(x) + v_i, \]

where \( L_i \) is an isometry of \( \mathbb{R}^d \). \( L_i \geq 1 \) is a scaling factor, \( v_i \in \mathbb{R}^d \) (for future calculations we shall assume, that \( v_0 = 0 \)). There exists the unique nonempty compact set \( K^{(0)} \) such that

\[ K^{(0)} = \bigcup_{i=1}^M \Xi_i(K^{(0)}). \]

The set \( K^{(0)} \) is the (bounded) fractal.

Essential fixed points

The fixed point \( x \in K^{(0)} \) is an essential fixed point if there exists another fixed point \( y \in K^{(0)} \) and similitudes \( \Phi_i, \Psi_i \) such that \( \Phi_i(x) = \Psi_i(y) \). \( v^{(0)} \) is the set of the essential fixed points.

Nested fractals

The nested fractal is a fractal satisfying the following conditions:

1. \( \# \{v_i^{(0)}\} \geq 2 \)
2. There exists an open set \( U \subset \mathbb{R}^d \) such that for \( i \neq j \) \( \Phi_i(U) \cap \Phi_j(U) = \emptyset \) and \( U \backslash \Phi_i(U) \subset U \)
3. (Nesting) Let \( T, S \) be different 1-complexes. Then \( T \cap S = V(T) \cap V(S) \).
4. (Symmetry) For \( x, y \in V^{(0)} \) let \( R_{xy} \) denote the symmetry with respect to hyperplane bisecting the segment \( [x, y] \). Then \( \forall x, y \in V^{(0)} \) \( R_{xy} \) is the reflection on the set \( \{v_i^{(0)}\} \) we define graph structure \( E_i \) as follows:
   - \( (x, y) \in E_i \) if \( x \) and \( y \) are in the same 1-complex.
   - Then the graph \( (v_i^{(0)}, E_i) \) is connected.

Unbounded fractal

We define the unbounded fractal as

\[ K^{(\infty)} = \bigcup_{n=0}^\infty L^nK^{(0)}. \]

Figure: Lindstrøm snowflake: 7 similitudes, but only 6 essential fixed points.

Figure: Sierpiński Triangle: 3 similitudes, \( L = 3 \) (7 fixed points, but only 6 essential fixed points).

Figure: Sierpinski Pentagon: 5 similitudes, \( L = \frac{5}{2} \)

Figure: Sierpinski Triangle: 3 similitudes, \( L = 2 \)

Figure: An example of a fractal with three essential fixed points (description of one direction).

Figure: An example of a fractal with four essential fixed points (description of one direction).

Figure: An example of a fractal with five essential fixed points (description of one direction).

Figure: An example of a fractal with six essential fixed points and six fixed points - complexes creating ring structure (after three iterations).

Figure: An example of a fractal with six essential fixed points and six fixed points - complexes creating ring structure (after five iterations).

Shape of the complexes

Preposition. If \( d = 2 \), then the points from \( V^{(0)}_0 \) are vertices of a regular polygon.

Preposition. If \( d = 3 \), then the points from \( V^{(0)}_0 \) are vertices of a platonic solid.

Sufficient conditions on \( \mathbb{R}^d \)

Condition 1. If there are 3 essential fixed points (complexes are triangular), the fractal can be labelled.

Condition 2. If there are 4 essential fixed points (complexes are square), the fractal can be labelled.

Condition 3. If there are \( p \) essential fixed points, \( p \) prime, the fractal can be labelled.

Condition 4. If the number of fixed points is equal to the number of essential fixed points (complexes form a ring structure), the fractal can be labelled.

Figure: Four fractions of expanding Sierpinski gasket.

Figure: Labeling of vertices of the Sierpinski gasket.

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Figure: A 3D fractal with four essential fixed points: The Sierpinski pyramid (after five iterations).

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Impossibility of labelling the Lindstrøm snowflake

Can we consistently label vertices of any nested fractal? No, there exist fractals which cannot be labelled, e.g. the Lindstrøm snowflake.

Having labelled vertices of the bottom left complex clockwise as \( a, b, c, d, e \) we know that the bottom right complex must have its left vertex labelled as \( e \). We can label other vertices of this complex clockwise or counter-clockwise. Either way, the middle complex has its two adjacent vertices labelled \( b \) and \( d \) or \( b \) and \( d \), as the order on even number of labels of vertices in this complex is not preserved.

Figure: Regular labeling of vertices of the Lindstrøm snowflake.

Figure: Irregular labeling of vertices of the Lindstrøm snowflake.

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Bibliography